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Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture

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Abstract: We conjecture that the discrete light-cone quantization (DLCQ) of strings on the maximally supersymmetric type IIB plane-wave background in the sector with J units of light-cone momentum is a supersymmetric $0+1$ dimensional $\mathrm{U}(J)$ gauge theory (quantum mechanics) with $\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)$ superalgebra. The conjectured hamiltonian for the plane-wave matrix (string) theory, the tiny graviton matrix theory, is the quantized (regularized) three brane action on the same background. We present some pieces of evidence for this conjecture through analysis of the hamiltonian , its vacua, spectrum and coupling constant. Moreover, we discuss an extension of our conjecture to the DLCQ of type IIB strings on $AdS_5\times S^5$ geometry.

Keywords: AdS-CFT and dS-CFT Correspondence, M(atrix) Theories, Penrose limit and pp-wave background.

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Contents

1. Introduction

According to Banks-Fischler-Susskind-Shenker (BFSS) conjecture discrete light-cone quantization (DLCQ) of M-theory in the sector with N units of light-cone momentum is described by a supersymmetric quantum mechanics, a $0+1$ dimensional super Yang-Mills $\mathrm{U}(N)$ gauge theory [1]. The BFSS hamiltonian is basically describing dynamics of N D0-branes. D0-branes from 11 dimensional viewpoint are gravity waves (gravitons) hence, according to BFSS proposal DLCQ of M-theory is described by N gravitons of the same theory. This conjecture so far has passed many crucial tests, for a review and more detailed discussions see [2]. The original BFSS hamiltonian was proposed to describe DLCQ of M-theory on the flat space background. This conjecture, however, was generalized to weakly curved backgrounds [3] and recently, by Berenstein-Maldacena-Nastase (BMN), to M-theory on the maximally supersymmetric 11 dimensional plane-wave background [4].

The BFSS Matrix model is based on the D0-brane dynamics. The same action can also be obtained from the regularized ("quantized") M2-brane action in the 11 dimensional flat space background, once the light-cone gauge is fixed [5]. The same idea, i.e. quantization of M2-branes in the 11 dimensional plane-wave background, was shown to lead to BMN Matrix model [6]. The BMN Matrix model has some specific features which makes it more tractable than the BFSS case: there are no flat directions and hence (at least for finite N) BMN Matrix model has a large set of normalizable supersymmetric vacua all of them have interpretation in terms of spherical M2-brane giant gravitons in the 11 dimensional plane-wave background [4, 6]. Giant gravitons, are spherically shaped D or M -branes, which are following light-like geodesics and the spatial part of their worldvolume is a sphere, blown up due to the existence of non-zero form flux in the background [7]. Intuitively, one may interpret the BMN Matrix model as a theory of N spherical M2-branes of very small size each of which carrying one unit of the light-cone momentum. For the reasons which will become apparent in the next section, these objects will be called (eleven dimensional) "tiny" gravitons, i.e. DLCQ of M-theory on the 11 dimensional plane-wave background is a theory of N tiny (membrane) gravitons.

In this paper, inspired by the BFSS idea, and in light of the above description of BMN matrix model, we present a conjecture for the DLCQ of type IIB string theory on the maximally supersymmetric ten dimensional plane-wave background. This background, which hereafter would be referred to as "the" plane-wave to distinguish it from other plane-wave geometries, as a solution of type IIb supergravity is given by

$$ds^2 = -2dX^+dX^- - \mu^2(X^iX^i + X^aX^a)(dX^+)^2 + dX^idX^i + dX^adX^a\,, \quad (1.1a)$$

$$F_{+ijkl} = \frac{4}{g_s}\mu\ \epsilon_{ijkl}\,, \qquad F_{+abcd} = \frac{4}{g_s}\mu\ \epsilon_{abcd}\,, \quad (1.1b)$$

where $i, a = 1, 2, 3, 4$. The above (self-dual) fiveform field is the field strength of the fourform C_4 ($F = dC_4$)

$$C_{+ijk} = -\frac{\mu}{g_s}\epsilon_{ijkl}X^l\,, \qquad C_{+abc} = -\frac{\mu}{g_s}\epsilon_{abcd}X^d\,. \quad (1.2)$$

Here we follow conventions and notations of [8]. This background has a globally defined light-like Killing vector $\partial/\partial X^-$. String theory σ-model on this background, in the light-cone gauge, is shown to be solvable [9]. And again according to BMN [4], as a specific case (or extension of) the usual AdS/CFT duality [10], type IIB string theory on the plane-wave background has been conjectured to be dual to a specific subsector, known as BMN sector, of $\mathcal{N} = 4$, $D = 4$ U(N) SYM gauge theory. The BMN sector consists of "almost" chiral primary operators with large R-charge J, $J \sim \sqrt{N}$, and by almost chiral-primary we mean operators of conformal dimension Δ and R-charge J where $\Delta - J \ll J$. For more detailed discussion on the plane-wave/SYM duality see [8, 11].

The plane-wave (1.1) has a one dimensional light-like boundary [12] and one might wonder whether besides the BMN description, similarly to the AdS/CFT case [13], we have a holographic description for strings on the plane-wave background.[1]

This holographic description, if it exists, should then be a $0+1$ dimensional theory, presumably a gauge theory. The goal of this paper is to propose a possible candidate for such a holographic formulation. For some previous attempts in this direction see [12, 14]. Noting that the boundary of the plane-wave is light-like, this holographic description for the sector of a given light-cone momentum is then expected to provide us with the DLCQ of strings on the type IIB plane-wave background.

The paper is organized as follows. In section 2, after introducing the "tiny" gravitons and some suggestive arguments, we present the statement of our conjecture and the proposed light-cone hamiltonian for the DLCQ of strings on the plane-wave background. In section 2.1 we present the condition (the Gauss law constraint) that all physical states of the Matrix model should satisfy. In section 2.2 we discuss the limit under which we recover the type IIB string theory. In section 2.3 we discuss the extension of our conjecture to the DLCQ of type IIB strings on $AdS_5 \times S^5$ geometry. In section 3, we present some pieces of evidence in support of our proposal. These include analysis of the vacua of the theory and the spectrum about these vacua. We close with conclusions and discussions. In the four Appendices we have gathered necessary technical points. In appendix A, we present the light-cone hamiltonian for a 3-brane in the plane-wave background. In appendix B, we briefly review Nambu brackets and their quantization. In particular we present a novel solution to the problem of quantization of Nambu odd brackets. In appendix C, we have presented the superalgebra of the plane-wave background and its representation in terms of matrices of our tiny graviton Matrix theory. Appendix D contains a brief review on fuzzy spheres, and in particular fuzzy three sphere. In this appendix we also present a novel and unified way for fuzzifying generic d spheres.

2. The proposal

Our proposal is that the DLCQ of strings on the ten dimensional plane-wave background in the sector with J units of light-cone momentum, similarly to the BMN Matrix model case (cf. discussions of Introduction), is described by the dynamics of J "tiny" gravitons, which are very small size 3-branes, on this background. In order to motivate and explain our proposal let us recall the giant gravitons in $AdS_5 \times S^5$ geometry. Following [7], consider a 3-brane moving along a light-like geodesic in the S^5 with angular momentum J. In two of the three minimum energy, half BPS configurations (corresponding to the giant gravitons

[1] The plane-wave geometry (1.1) can be obtained from $AdS_5 \times S^5$ solution upon taking the Penrose limit [4]. However, the causal structure, the Penrose diagram and hence the boundary of the plane-wave are different than $AdS_5 \times S^5$.

grown in S^5 or in AdS_5 [15]) 3-brane worldvolume is a three sphere of radius

$$\left(\frac{R_{\rm giant}}{R_{AdS}}\right)^2 = \frac{J}{N}\,, \tag{2.1}$$

where N is the number of fiveform fluxes on S^5. Next recall that in the $AdS_5\times S^5$ geometry,

$$\left(\frac{R_{AdS}}{l_s}\right)^4 = g_s N\,. \tag{2.2}$$

Noting that ten dimensional Planck length $l_{\rm Pl}$ and the string scale l_s are related as $l_{\rm Pl}^4 = l_s^4 g_s$, we have

$$\left(\frac{R_{AdS}}{l_{\rm Pl}}\right)^4 = N\,. \tag{2.3}$$

Therefore, the size of the smallest giant graviton, which will henceforth be called a *tiny* graviton, i.e. the case in which the 3-brane is carrying one unit of angular momentum $J=1$, is given by[2]

$$R_{\rm tiny} = \frac{l_{\rm Pl}^2}{R_{AdS}}\,. \tag{2.4}$$

This is a very remarkable result: for the cases where we can trust the supergravity description, $R_{AdS}\gg l_{\rm Pl}$, $R_{\rm tiny}\ll l_{\rm Pl}$ and hence we would expect the fundamental description of the theory should come from the *tiny* gravitons.[3] Eq. (2.4) also justifies why we have called these objects tiny gravitons.

The tiny gravitons, being 3-branes of small size, should show a brane-like behaviour. Namely, we expect that when J number of them sit on top of each other we should have a U(J) gauge theory description. This result about the enhancement of the gauge symmetry, although non-trivial, has been supported by studying BPS open string-type states in the giant graviton worldvolume theory [16]. In this respect tiny gravitons are similar to D0-branes of BFSS and may be used to obtain a (non-perturbative) DLCQ formulation for string theory on $AdS_5\times S^5$ or the corresponding plane-wave.

To obtain the action for J tiny gravitons, we follow the logic of [5] where the corresponding Matrix model is obtained as a regularized (quantized) version of M2-brane light-cone hamiltonian, but now for 3-branes. In other words, we conjecture that DLCQ of type IIB strings on the plane-wave background (1.1) is nothing but a quantized 3-brane theory.

The 3-brane light-cone hamiltonian for the plane-wave background is given in appendix A, in eqs. (A.3) and (A.7). Since we are looking for a DLCQ description, we need to compactify X^- on a circle of radius R_-:

$$X^- \equiv X^- + 2\pi R_-\,. \tag{2.6}$$

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[2] One should note that the notion of size for giant gravitons we have been using so far, like the giant graviton themselves, is a classical one. Indeed, the main idea and subject of the paper is how to quantize them.

[3] These results may be repeated for Mp tiny gravitons, $p=2,5$ in $AdS_{4,7}\times S^{7,4}$ or the corresponding 11 dimensional plane-wave background. The analog of (2.4) for $AdS_q\times S^{D-q}$ space (D=11) is

$$R_{\rm tiny}^{p-1} R_{AdS}^{q-p} = l_{\rm Pl}^{q-1}\,. \tag{2.5}$$

Note that now $l_{\rm Pl}$ is the 11 dimensional Planck length. It is amusing that for $p=3$ (2.5) reproduces (2.4).

This leads to the quantization of the light-cone momentum p^+,

$$p^+ = \frac{J}{R_-}\,. \tag{2.7}$$

In our conventions μ and R_- have dimension of energy, p^+ has dimension length and hence μp^+ is dimensionless.[4]

To quantize the 3-brane theory, following [2, 5, 6], we prescribe: replace $X^I(\sigma)$, $P(\sigma)_I$, $\psi_{\alpha\beta}(\sigma)$, $\psi_{\alpha\beta}(\sigma)$ with $J \times J$ matrices, i.e.

$$\begin{aligned} X^I(\sigma) &\longleftrightarrow X^I_{Mat} \\ P^I(\sigma) &\longleftrightarrow J\Pi^I_{\rm Mat} \\ \psi(\sigma) &\longleftrightarrow \sqrt{J}\psi_{\rm Mat}\,, \end{aligned} \tag{2.8}$$

together with

$$\frac{1}{p^+}\int d^3\sigma\, * \longleftrightarrow R_-{\rm Tr}\ \ * \tag{2.9}$$

$$\{F,G,K\} \longleftrightarrow J\ [F,G,K,\mathcal{L}_5]\,. \tag{2.10}$$

In the above, and in what follows, we will use I, J indices, $I, J = 1, 2, \ldots, 8$ to denote the eight transverse directions, i.e. $X^I = (X^i, X^a)$. For the definition of quantized Nambu bracket see appendix B and for the definition of the $J \times J$ matrix $\mathcal{L}_5$ see appendix D, eqs. (D.8) and (D.9). After the above replacements we obtain the tiny graviton Matrix theory hamiltonian which we conjecture to give the non-perturbative description of strings on the plane-wave in the sector with J units of light-cone momentum. Explicitly, the statement of our conjecture is:

Conjecture 1. *The theory of J tiny gravitons, which is a* U(J) *supersymmetric quantum mechanics with the* PSU(2|2) × PSU(2|2) × U(1) *symmetry, is the Matrix theory describing the DLCQ of string on plane-waves with light-cone momentum $p^+ = J/R_-$, R_- is the light-like compactification radius. The hamiltonian of this Matrix model is:*

$$\begin{aligned} \mathbf{H} = R_-{\rm Tr}\Bigg[&\frac{1}{2}\Pi_I^2 + \frac{1}{2}\left(\frac{\mu}{R_-}\right)^2 X_I^2 + \frac{1}{2\cdot 3! g_s^2}[X^I,X^J,X^K,\mathcal{L}_5][X^I,X^J,X^K,\mathcal{L}_5] - \\ &- \frac{\mu}{3! R_- g_s}\left(\epsilon^{ijkl}X^i[X^j,X^k,X^l,\mathcal{L}_5] + \epsilon^{abcd}X^a[X^b,X^c,X^d,\mathcal{L}_5]\right) + \\ &+ \left(\frac{\mu}{R_-}\right)\left(\psi^{\dagger\alpha\beta}\psi_{\alpha\beta} - \psi_{\dot\alpha\dot\beta}\psi^{\dagger\dot\alpha\dot\beta}\right) + \\ &+ \frac{2}{g_s}\left(\psi^{\dagger\alpha\beta}(\sigma^{ij})_\alpha^{\ \delta}[X^i,X^j,\psi_{\delta\beta},\mathcal{L}_5] + \psi^{\dagger\alpha\beta}(\sigma^{ab})_\alpha^{\ \delta}\,[X^a,X^b,\psi_{\delta\beta},\mathcal{L}_5]\right) - \\ &- \frac{2}{g_s}\left(\psi_{\dot\delta\dot\beta}(\sigma^{ij})_{\dot\alpha}^{\ \dot\delta}\,[X^i,X^j,\psi^{\dagger\dot\alpha\dot\beta},\mathcal{L}_5] + \psi_{\dot\delta\dot\beta}(\sigma^{ab})_{\dot\alpha}^{\ \dot\delta}\,[X^a,X^b,\psi^{\dagger\dot\alpha\dot\beta},\mathcal{L}_5]\right)\Bigg]\,. \end{aligned} \tag{2.11}$$

[4] Note that we have set $\alpha' = l_s^2 = 1$. Recovering the powers of α', $\mu p^+\alpha'$ is the dimensionless, physical light-cone momentum.

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One of the advantages of our proposal (e.g. compared to that of [14]) is that it explicitly exhibits the invariance under the PSU(2|2) $\times$ PSU(2|2) $\times$ U(1) superalgebra, which is the superalgebra of the plane-wave background; see appendix C for the superalgebra and its representation in terms of the $J \times J$ matrices. The other advantage is that, similarly to BMN Matrix model [4, 6], there are no flat directions and the flat directions are lifted by the mass terms coming form the background plane-wave metric.

The U(J) gauge symmetry of the above hamiltonian is in fact a discretized (quantized) form of the spatial diffeomorphisms of the 3-brane. As is evident from the above construction (and the discussions of the Appendices) we expect in $J \to \infty$ limit to recover the diffeomorphisms. In this respect, it is very similar to the usual BFSS Matrix model in which the gauge symmetry is the regularized form of the diffeomorphisms on the membrane worldvolume [5]. The quantization of diffeomorphisms on a three dimensional surface (in our case a three sphere), unlike the two dimensional surfaces corresponding to membranes, with a desirable continuum limit is a formidable task. The fuzzy sphere S^3_F, however, provides us with a unique solution (cf. appendix D). Moreover, the above proposal closely parallels a similar conjecture about M-theory on the 11 dimensional plane-wave background, the BMN Matrix model, where tiny membrane gravitons on the 11 dimensional plane-wave background are the fundamental objects of BMN Matrix theory [6].

It is worth noting that in the classical hamiltonian we started with, (A.3), we have not included the gauge field in the original Born-Infeld action and as such one may worry that we have not included all the necessary degrees of freedom to begin with. As we will show in section 3.2.1, these U(1) gauge fields, and the corresponding photon states are indeed already accounted for and they appear in the spectrum of theory once we expand the theory about its vacua.

2.1 Physical states

The hamiltonian (2.11) can be obtained from a $0+1$ dimensional U(J) gauge theory lagrangian, in the temporal gauge. Explicitly, the only component of the gauge field, A_0, has been set to zero. To ensure the $A_0 = 0$ gauge condition, all of our physical states must satisfy the Gauss law constraint arising from equations of motion of A_0. Similarly to the BFSS [1] and BMN [6] cases, these constraints, which consists of $J^2 - 1$ independent conditions are:

$$\left(i[X^i, \Pi^i] + i[X^a, \Pi^a] + 2\psi^{\dagger \alpha\beta}\psi_{\alpha\beta} + 2\psi^{\dagger \dot\alpha\dot\beta}\psi_{\dot\alpha\dot\beta} \right) |\phi\rangle_{\rm phys} = 0 \,. \tag{2.12}$$

These constraints are the requirement of SU(J) invariance of the physical states.

One may trace back the Gauss law constraint (2.12) to the continuum 3-brane action. For this, recall that in the procedure of fixing the light-cone gauge we need to impose constraints (A.6). (These constraints in the string theory language give rise to the level matching condition on the closed string states [18].) Upon quantization, and adding the fermionic terms, (A.6) goes over to the Gauss law constraint (2.12) which should be imposed on the physical states.

2.2 String theory limit

The hamiltonian (2.11) is proposed to describe type IIB string theory on the plane-wave with compact X^- direction. The "string theory limit" is then a limit where R_- is taken to infinity, keeping p^+ fixed, i.e.

$$J, R_- \to \infty\,, \qquad \mu\,, \quad p^+ = \frac{J}{R_-}\,, \quad g_s \qquad \text{fixed}\,. \tag{2.13}$$

In fact one can show that in the above string theory limit one can rescale X's such that μ, p^+ only appear in the combination μp^+. Therefore the only parameters of the continuum theory are μp^+ and g_s. The plane-wave that we obtain after the Penrose limit, and hence the usual BMN double scaling limit, would appear in our model after taking the string theory limit.

One of the questions that often arise in the DLCQ descriptions of flat space (and in particular the BFSS Matrix model) is whether we can recover a covariant theory in the continuum limit. Here we briefly discuss this question posed for our tiny graviton Matrix theory. The first point we would like to remind the reader is an important difference between plane-wave background and the flat space. For concreteness let us compare the isometries of the ten dimensional plane-wave (1.1) and a ten dimensional flat space. The plane-wave metric has 30 isometries, 14 of which are $\mathrm{SO}(4) \times \mathrm{SO}(4)$ rotations and translations along X^+, X^-. The other 16 are forming a Heisenberg-type algebra, in particular we note that generator of translations along X^-, p^+, commutes with all the isometries and the light-cone boosts, J^{+-}, J^{I-} are all absent [8]. This should be contrasted with the flat space Poincare group which is $10 + 45 = 55$ dimensional, including light-like boosts J^{+-}, J^{I-}. Therefore, in the above formulation of the plane-wave string theory, which makes all the "dynamical" and "kinematical" isometries manifest, the DLCQ description is already a complete one. (For the definition of kinematical and dynamical isometries, which parallels the same terminology about supercharges see [8]). In fact one can show that the kinematical isometries, similarly to the BMN case [6], can be represented in terms of the U(1) part (trace part) of U(J) matrices.

2.3 Extension of the proposal to DLCQ of strings on $AdS_5 \times S^5$

In this section we extend our conjecture to beyond the Penrose limit, to the full $AdS_5 \times S^5$. The proposal is that the hamiltonian for DLCQ of strings on $AdS_5 \times S^5$ with N units of RR flux on S^5, is the same as (2.11) with $R_- = \sqrt{g_s N}$. In this case the gauge group can be as large as U(N). The string theory limit (2.13) is then equivalent to large N ($N \to \infty$) limit.

Since we are looking for a DLCQ description we need to consider light-like geodesics in $AdS_5 \times S^5$. There are two kinds of them: those along the radial direction in the AdS_5 and the ones which are along a circle inside S^5. The former geodesics hit the boundary, moreover it is not possible to compactify the radial direction on a circle. So, for the purpose of DLCQ we are only left with the latter, geodesics in S^5. The objects which follow such geodesics are gravitons of various size, giant, normal or tiny gravitons. In fact sitting in the rest frame of the giant (or tiny) gravitons, from the viewpoint of the observer who uses the usual $AdS_5 \times S^5$ global time as time coordinate, is like boosting to the infinite momentum frame, in which the strings and giants are made out of tiny graviton "partons". Next, we note that our earlier discussion in the opening of this section and in particular (2.4) holds for the $AdS_5 \times S^5$.

The above proposal, if correct, would shed light on the stringy exclusion principle [7]. It would also be very desirable to rederive or confirm the above proposal from the usual $\mathcal{N} = 4$ CFT description. We will comment more on this proposal in the discussion section, however, it needs a thorough and careful analysis. We hope to address this in a future work [17].

3. Evidence for the proposal

In this section we present some evidence and arguments in support of our conjecture some of which we have already discussed. First, we note that by construction our $0+1$ dimensional gauge theory is supersymmetric one with the PSU(2|2) × PSU(2|2) × U(1) superalgebra, which is the superalgebra of the plane-wave background. (We emphasize that this gauge theory is *not* a Yang-Mills theory). Furthermore, generically supersymmetric quantum mechanics with 16 supercharges are uniquely determined once we specify superalgebra and the gauge group. Hence, if the type IIB string on the plane-wave background admits a holographic description our hamiltonian is the first viable candidate. Indeed confirmation of the uniqueness of the U(J) quantum mechanics with PSU(2|2) × PSU(2|2) × U(1) supersymmetry, if we believe in holography, can be regarded as a proof for our proposal [17]. To see the second piece of evidence we work out the vacua and zero energy configurations.

3.1 Vacua and zero energy solutions

Inspired by the analysis of the continuum case [16], we note that for the static ($\Pi = 0$), bosonic ($\psi = 0$) configurations the hamiltonian takes the form

$$\begin{aligned} V = R_- \mathrm{Tr}\Bigg[&\frac{1}{2}\left(\frac{\mu}{R_-}X^l + \frac{1}{3! g_s}\epsilon^{ijkl}[X^i, X^j, X^k, \mathcal{L}_5]\right)^2 + \\ &+\frac{1}{2}\left(\frac{\mu}{R_-}X^d + \frac{1}{3! g_s}\epsilon^{abcd}[X^a, X^b, X^c, \mathcal{L}_5]\right)^2 + \\ &+\frac{1}{4g_s^2}\Big([X^i, X^j, X^a, \mathcal{L}_5][X^i, X^j, X^a, \mathcal{L}_5] + [X^i, X^a, X^b, \mathcal{L}_5][X^i, X^a, X^b, \mathcal{L}_5]\Big)\Bigg]. \end{aligned} \tag{3.1}$$

Evidently, there are three types of zero energy vacuum configurations:

$$X^a = 0\,, \qquad [X^i, X^j, X^k, \mathcal{L}_5] = -\frac{\mu g_s}{R_-}\epsilon^{ijkl}X^l \tag{3.2a}$$

$$X^i = 0\,, \qquad [X^a, X^b, X^c, \mathcal{L}_5] = -\frac{\mu g_s}{R_-}\epsilon^{abcd}X^d \tag{3.2b}$$

$$X^i = X^a = 0\,. \tag{3.2c}$$

The physics of the first two solutions, (3.2a) and (3.2b), are very similar, because there is a Z_2 symmetry of the background plane-wave under which $X^i \leftrightarrow X^a$. Of course if we choose to expand the theory about either of these vacua, this Z_2 symmetry is spontaneously broken. Therefore, we only focus on solutions of (3.2a) and (3.2c).

Solutions of (3.2a). Solutions of (3.2a) can be classified in terms of (irreducible or reducible) representations of Spin(4) and in general they are of the form of concentric fuzzy three spheres of various radii (cf. appendix D). (This is very similar to the case of BMN Matrix theory studied in detail in [6].) The size of each three sphere is given by the fraction of the total light-cone momentum J that it carries (where radius squared $\propto$ fraction of light-cone momentum) and the sum of radii squared of these fuzzy spheres is $\frac{\mu g_s}{R_-}J$.

Here we only focus on the solution given by irreducible Spin(4) representation which corresponds to a single S^3_F. The reducible representations and their detailed analysis is postponed to future works. Noting (D.5) and (D.12), the radius of this S^3_F, in units of l_s, is

$$R^2 = \frac{\mu g_s}{R_-}J = \mu p^+ g_s \tag{3.3}$$

with the fuzziness l:

$$l^2 = \frac{\mu g_s}{R_-}l_s^2\,. \tag{3.4}$$

(Note that in (3.4) we have reintroduced factors of l_s.) In the second equality of (3.3) we have used definition of p^+ (2.7). This irreducible solution, which will be denoted by $X = J$ vacuum, is indeed the giant graviton of [7, 16]. In other words, our tiny graviton Matrix theory contains giant gravitons as zero energy configurations. In our picture a giant graviton of radius R is a state in which J tiny gravitons are blown up into a three sphere.

Here we would like to compare (3.3) to results of [7, 16]. As we see in both cases, i.e. (2.1) and (3.3), R^2 is proportional to J. We would like to present this fact as the second evidence in support of our proposal. This point, which may seem a trivial result of our construction, is very remarkable. Note that (2.1) is coming from a physical condition, namely stability of the spherical brane (and in general for a p sphere giant $p = 2, 3, 5$ $R^{p-1} \propto J$ [7]), whereas (3.3) is a result of Spin(4) = SU(2) $\times$ SU(2) group theory and its representations.[5] This result becomes less trivial noting that, unlike the membrane case, there is no unique prescription for quantization of diffeomorphisms of a three dimensional surface. In this regard (3.3) is a confirmation of the fact that our quantization proposal, namely replacing a three sphere giant with a fuzzy three sphere, is the right one.

The (3.2c) solution, $X = 0$ vacuum. The physical interpretation of (3.2c) solution, the $X = 0$ vacuum, is quite non-trivial; as the second part of our proposal we conjecture that:

Conjecture 2. *In the string theory limit discussed in section 2.2, this vacuum quantum mechanically becomes the vacuum for strings on the plane-wave background, i.e. the BMN vacuum [4]. The fundamental type IIB closed strings, however, appear as non-perturbative objects in this vacuum.*

[5] Of course a similar relation also holds for spherical M2-branes [6], while is not true for M5 giants. We will comment more on this point in section 4.

As first evidence for the above proposal, we note that:

i) all of these vacua, reducible or irreducible are half BPS states. That is, they preserve all the dynamical supercharges. To see this recall the superalgebra (C.3) and the fact that all of these vacua are zero energy solution with $\mathbf{J}_{ij} = \mathbf{J}_{ab} = 0$.

ii) In string theory limit (2.13), among the (infinitely many) vacua, reducible or irreducible, we only remain with $X = 0$ and two irreducible, single giant $X = J$ vacua. In the BMN gauge theory, these states should correspond to chiral-primary operators [4] and the only single particle states available are either single string (BMN vacuum) or single giant graviton vacua.[6] We would like to emphasize that the $X = 0$ vacuum classically does not look like a string theory vacuum state, but would become BMN vacuum quantum mechanically. In other words, we propose that the 't Hooft strings of our $\mathrm{U}(J)$ gauge theory are indeed type IIB strings on the plane-wave background. Our proposal for the $X = 0$ vacuum has a counterpart in the BMN Matrix model where the $X = 0$ (membrane) vacuum quantum mechanically behaves as a single five brane vacuum [19]. In section 3.2.2 we will provide another piece of evidence in support of the above proposal.

In the BMN literature there have been two other quantum mechanical proposals for *perturbative* dynamics of fundamental type IIB closed strings. The first is obtained by summarizing the information coming from the $\mathcal{N} = 4, D = 4$ gauge theory calculations in the BMN sector, e.g. see [11, 22]. The second one is the Verlinde's String Bit model [23]. We will comment on the possible relation between our $\mathrm{U}(J)$ tiny graviton Matrix theory and the string bit model in section 4.

[6] Some brief comments on the connection to the usual $\mathcal{N} = 4, D = 4$ gauge theory are in order: denote one of the three complex scalars of $\mathcal{N} = 4, D = 4, \mathrm{U}(N)$ SYM theory by Z, the chiral-primary operators are then made out of gauge invariant products of Z's. Let us focus on chiral-primaries with R-charge J (i.e. they are composed of J number of Z's.) The gauge invariant operators are then obtained by summing over all $\mathrm{U}(N)$ indices. There are three different ways of constructing such gauge invariant operators: using trace, (sub)determinant [20] and Schur's polynomials [21]. In the BMN limit, however, the latter two would essentially become identical. We then remain with the two following possibilities to expand a chiral-primary operator of R-charge J:

$$\text{Trace basis :} \qquad \mathrm{Tr}Z^J, \ \ :\mathrm{Tr}Z^{J_1}\mathrm{Tr}Z^{J-J_1}:, \ \ \ldots, \ \ :(\mathrm{Tr}Z)^J: \tag{3.5a}$$

$$\text{subdeterminant basis :} \qquad \mathcal{O}_J\ , \ \ :\mathcal{O}_{J_1}\mathcal{O}_{J-J_1}:, \ \ \cdots, \ \ :\mathcal{O}_1^J: \tag{3.5b}$$

with

$$\mathcal{O}_k = \mathcal{N}_k \epsilon_{i_1 i_2 \cdots i_k i_{k+1} i_{k+2} \cdots i_N} \epsilon^{j_1 j_2 \cdots j_k j_{k+1} j_{k+2} \cdots j_N} Z^{j_1}_{i_1} Z^{j_2}_{i_2} \cdots Z^{j_k}_{i_k} \tag{3.6}$$

where $\mathcal{N}_k^{-2} = k!(N-k)!$ is the normalization factor and i, j indices run from one to N. According to BMN $\mathrm{Tr}Z^J$ corresponds to perturbative single string vacuum [4] while $\mathcal{O}_J$ is proposed to describe a single giant graviton state [20]. In the same spirit n-trace (or n-subdeterminant) operators of (3.5a) (or (3.5b)) corresponds to an n-strings (or n-giant gravitons) vacuum state. Next we note that the last two operators in the row, $:(\mathrm{Tr}Z)^J:$, which according to BMN picture corresponds to J string vacuum, and $:\mathcal{O}_1^J:$, which corresponds to the J tiny giant gravitons, are exactly equal (note that $\mathcal{O}_1 = \mathrm{Tr}Z$). This observation may help with a better understanding of the $X = 0$ vacuum.

3.2 Spectrum of the theory about the vacua

In this section we start with studying the tiny graviton Matrix theory by performing a perturbative expansion about its vacua. Here we only consider $X = J$ and $X = 0$ vacua and the analysis of rest of the vacua (which are in reducible representations of Spin(4)) are postponed to future works. The purpose of choosing these two vacua among the others is that, for $X = J$, which in string theory limit (2.13) should reproduce the single giant graviton state, the spectrum should match with the one given in [16]. This would provide us with another check for our conjecture. In particular, there is a non-trivial, crucial test that our conjecture should pass: As we mentioned earlier, to obtain the hamiltonian, we started form the Born-Infeld action (A.1) in which we have not included the U(1) gauge fields of the 3-brane. So, it is very important for the consistency of our theory to make sure that the photon states corresponding to this gauge field are present in the spectrum of our theory about $X = J$ vacuum. Although this sounds too good to be true, performing explicit and detailed calculation we confirm the presence of these photon modes.

The reason to study the spectrum of the theory about the $X = 0$ vacuum is to provide another piece of evidence in support of the conjecture we made in section 3.1. As we proposed fundamental type IIB closed strings should be realized as non-perturbative states about this vacuum. (This point would become clearer in section 3.3 where we show that the effective coupling about this vacuum goes to infinity in the the string theory limit.) On the other hand, only the information about BPS states which are protected by supersymmetry can be trusted at strong coupling limit. As we will show the spectrum of BPS states about this vacuum exactly matches with the spectrum of BPS type IIB closed string states in the plane-wave background.

3.2.1 Irreducible vacuum

In order to study the theory about the irreducible solution of (3.2a), we expand X^i's as

$$X^i = \sqrt{\frac{\mu g_s}{R_-}} J^i + Y^i \,, \tag{3.7}$$

where J^i's are in irreducible $J \times J$ representations of Spin(4) which satisfy

$$[J^j, J^k, J^l, \mathcal{L}_5] = \epsilon^{ijkl} J^i \,, \tag{3.8}$$

and Y^i's parametrize perturbations about this vacuum. Inserting (3.7) into (3.1) and keeping the terms second order in Y^i and X^a we have:

$$V^{(2)}_{X=J} = \frac{\mu^2}{2R_-} \operatorname{Tr}\left[\left(Y^l + \frac{1}{2}\epsilon^{ijkl}[J^i, J^j, Y^k, \mathcal{L}_5]\right)^2 + X_a^2 + \frac{1}{2}[J^i, J^j, X^a, \mathcal{L}_5][J^i, J^j, X^a, \mathcal{L}_5]\right]. \tag{3.9}$$

Spectrum of X^a modes: using the by-parts integration property of the four brackets (cf. appendix B) the last term of (3.9) can be written as

$$\frac{1}{2}\operatorname{Tr}\left(X^a \left[J^i, J^j, [J^i, J^j, X^a, \mathcal{L}_5], \mathcal{L}_5\right]\right).$$

Recalling the definition of J^i, namely (3.8) and also (D.8), it is straightforward to observe that the operator

$$\frac{1}{2}\left[J^i, J^j, \mathcal{L}_5, [J^i, J^j, \mathcal{L}_5], \star\right] \tag{3.10}$$

is in fact the second rank Casimir of Spin(4) = SU(2) × SU(2), and hence

$$[J^i, J^j, \mathcal{L}_5, [J^i, J^j, \mathcal{L}_5], X^a_j]] = 2j(j+2)X^a_j\,, \tag{3.11}$$

where X^a_j is the symmetric traceless rank j-tensor of Spin(4), namely it is in $(\frac{j}{2}, \frac{j}{2})$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $0 \leq j \leq n$, where n and J are related as given in (D.11). Using (3.11) one can read the eigen-frequencies of X^a_j, modes:

$$\omega_j^2 = \mu^2\left[1 + \frac{1}{2}\cdot 2j(j+2)\right] = \mu^2(j+1)^2 \qquad 0 \leq j \leq n\,. \tag{3.12}$$

Degeneracy of each of X^a_j modes is $(j+1)^2$. In particular note that $j = 0$ mode, with $\omega = \mu$, corresponds to center of mass motion of the giant three sphere.

Spectrum of Y^i modes and photon states: to diagonalize Y^i part of (3.9), consider the following eigenvalue equation:

$$\frac{1}{2}\epsilon^{ijkl}[J^i, J^j, Y^k, \mathcal{L}_5] = \lambda Y^l\,. \tag{3.13}$$

Solutions to the above equation are eigenmodes of (3.9) with eigen-frequencies $\omega^2 = \mu^2(1+\lambda)^2$ or

$$\omega = \mu|1+\lambda|\,. \tag{3.14}$$

To solve (3.13), we recall that $\frac{1}{2}\epsilon^{ijkl}[J^i, J^j, \mathcal{L}_5, \star]$ is indeed generator of SO(4) rotations in $J \times J$ representation. (cf. [16, section 2]). Inspired by the continuum solutions, one can check that (3.13) has three class of solutions:

i) $\hat{Y}_j$-modes: for which $\lambda = j, \qquad 0 \leq j \leq n$. The $\hat{Y}_j$ modes are in $(\frac{j+1}{2}, \frac{j+1}{2})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and have frequencies $\omega = \mu(j+1)$. Degeneracy of these modes is $(j+2)^2$.

ii) $\tilde{Y}_j$-modes: for which $\lambda = -(j+2), \qquad 1 \leq j \leq n$. The $\tilde{Y}_i$ modes are in $(\frac{j-1}{2}, \frac{j-1}{2})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with frequencies $\omega = \mu(j+1)$. $\tilde{Y}_i$ modes are j^2-fold degenerate.

iii) Zero modes, modes with $\lambda = -1$. These modes are in $(\frac{j-1}{2}, \frac{j+1}{2}) \oplus (\frac{j+1}{2}, \frac{j-1}{2})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with $1 \leq j \leq n$ and have $\omega = 0$. These modes come with degeneracy $2j(j+2)$ and are gauge degrees of freedom (corresponding to the SO(4) rotations of the fuzzy three sphere, cf. [16, section 2] and also [6, section 5]).

The physical modes, $\hat{Y}_j$ and $\tilde{Y}_j$ modes, both have the same frequency:

$$\omega_j = \mu(j+1)\,. \tag{3.15}$$

Mode	$SU(2) \times SU(2)$ rep.	Frequency	Degeneracy	Range
$\hat{Y}_j$	$(\frac{j+1}{2}, \frac{j+1}{2})$	$\mu(j+1)$	$(j+2)^2$	$0 \leq j \leq n$
$\tilde{Y}_j$	$(\frac{j-1}{2}, \frac{j-1}{2})$	$\mu(j+1)$	j^2	$1 \leq j \leq n$
X_j^a	$(\frac{j}{2}, \frac{j}{2})$	$\mu(j+1)$	$4 \cdot (j+1)^2$	$0 \leq j \leq n$
A_j^+	$(\frac{j-1}{2}, \frac{j+1}{2})$	$\mu(j+1)$	$j(j+2)$	$1 \leq j \leq n$
A_j^-	$(\frac{j+1}{2}, \frac{j-1}{2})$	$\mu(j+1)$	$j(j+2)$	$1 \leq j \leq n$

Table 1: Some of the modes and masses for the irreducible $X = J$ vacuum.

Note, however that for $j = 0$ we only have $\hat{Y}$ mode. These modes, which come with degeneracy equal to four, correspond to the center of mass motion of the giant. The $\tilde{Y}_{j=1}$ mode, however, corresponds to breathing mode of the giant, the mode in which the giant maintains its spherical shape and only has radial fluctuations.

Beside the above eigenmodes, Y^i modes also have two other physical eigenmodes solving the equation

$$\frac{1}{2}\epsilon^{ijkl}[J^i, J^j, Y^k, \mathcal{L}_5] = \pm\lambda\mathcal{L}_5 Y^l\,. \tag{3.16}$$

The eigenmodes resulting from (3.16), which will be denoted by $A_j^{\pm}$, are in $(\frac{j-1}{2}, \frac{j+1}{2})$ and $(\frac{j+1}{2}, \frac{j-1}{2})$ representation of $SU(2) \times SU(2)$ with $1 \leq j \leq n$ (respectively corresponding to $+$ and $-$ sign in (3.16)). Both of these modes have frequencies

$$\omega = \mu(j+1), \qquad 1 \leq j \leq n\,. \tag{3.17}$$

$A_j^{\pm}$ modes exactly correspond to the two different polarization of photon states. This is compatible with the results of the continuum limit [16]. So, our model passes another crucial test. Note that these photon states have the same $SU(2) \times SU(2)$ representation as the zero modes.[7] Moreover, the photon state starts from $j = 1$, and hence, there is no photon mode with the same frequency as the center of mass modes, $\hat{Y}_{j=0}, X^a_{j=0}$. We have summarized the results of this subsection in table 1.

As we see we have $8 \cdot (j+1)^2$ modes of frequency $\mu(j+1)$. It is straightforward to observe that all the modes in Table 1 are physical, in the sense that they all satisfy the Gauss law constraint (2.12). These modes fall into the same $PSU(2|2) \times PSU(2|2) \times U(1)$ representation. In a similar manner, one may also work out spectrum of fermionic modes which we do not present it here. The detailed analysis of the spectrum and its superalgebra representations are postponed to future works. (A similar analysis for the BMN Matrix model can be found in [24].)

The number of the mode that we have listed in the Table 1 together with corresponding fermionic modes, is $\sum_{j=1}^{n} 16(j+1)^2$, which for large n grows like n^3. The size of our matrices are, however, $J^2 \sim n^4$. This means that besides the above modes there are many other modes which we have not discussed here. Studying the structure of our $J \times J$ matrices and

[7] One should note that, for the fuzzy three sphere case, unlike the S_F^2, the algebra of functions generated from coordinates, J^i's, is *not* covering the whole $J \times J$ matrices. In fact $Mat_C(J)$ contains them with multiplicity two [34]. So, it is not strange to have zero modes and photons in the same $SU(2) \times SU(2)$ representations.

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Mode	Energy	Degeneracy
$\|0\rangle$	0	1
$\mathrm{Tr}a_i^\dagger\|0\rangle$	μ	4
$\mathrm{Tr}a_a^\dagger\|0\rangle$	μ	4
$\mathrm{Tr}\psi^{\dagger\alpha\beta}\|0\rangle$	μ	4
$\mathrm{Tr}\psi^{\dagger\dot\alpha\dot\beta}\|0\rangle$	μ	4

Table 2: Lowest energy states about $X = 0$ vacuum.

how they are filled up with Spin(4) representations, suggests that (at least some of) these extra modes, which are in (j_1, j_2), $|j_1 - j_2| \geq 2$ of $SU(2) \times SU(2)$, are of the form of the spike solutions of [16], which in the continuum limit become the excitation modes of open strings ending on the giant and some of these mode would become massive and decouple in the contimuum limit. This of course deserves a more thorough study, postponed to future works.

3.2.2 $X = 0$ vacuum

In this case since the vacuum configuration is $X = 0$, the X^i and X^a's can directly be regarded as perturbations about this vacuum. The hamiltonian (2.11) up to second order in perturbations is

$$\mathbf{H}^{(2)}_{X=0} = \mathrm{Tr}\left[\left(\frac{R_-}{2}\Pi_i^2 + \frac{\mu^2}{2R_-}X_i^2\right) + \left(\frac{R_-}{2}\Pi_a^2 + \frac{\mu^2}{2R_-}X_a^2\right) + \mu\psi^{\dagger\alpha\beta}\psi_{\alpha\beta} - \mu\psi_{\dot\alpha\dot\beta}\psi^{\dagger\dot\alpha\dot\beta}\right]. \tag{3.18}$$

(3.18) is of the form of $16 = 2 \cdot 8$ decoupled harmonic oscillators, all with frequency μ. Hence, one can readily read the spectrum. However, let us first introduce the creation operators

$$a_i^\dagger = \sqrt{\frac{\mu}{2R_-}}X^i - i\sqrt{\frac{R_-}{2\mu}}\Pi_i\,, \qquad a_a^\dagger = \sqrt{\frac{\mu}{2R_-}}X^a - i\sqrt{\frac{R_-}{2\mu}}\Pi_a\,. \tag{3.19}$$

The lowest energy states about this vacuum are gathered in table 2. One may also construct two and higher oscillator states, e.g. $\mathrm{Tr}a_i^\dagger\mathrm{Tr}a_j^\dagger|0\rangle$ or $\left(\mathrm{Tr}(a_i^\dagger a_j^\dagger) - \frac{1}{J}\mathrm{Tr}a_i^\dagger\mathrm{Tr}a_j^\dagger\right)|0\rangle$ which have energy 2μ.

The important point here is that: the action (3.18) is basically the same as the plane-wave string theory σ-model action, once we ignore the string tension term (i.e. large μ limit). This in particular implies that the BPS states of strings on the plane-wave background, which necessarily have zero worldsheet momentum, have exactly the same spectrum as the one generated by (3.18). In other words (3.18) reproduces the spectrum of type IIb supergravity on the plane-wave background, this can be explicitly checked comparing spectrum of (3.18) with TABLE I of [8].

3.3 Effective coupling of the theory

In the previous section we studied the spectrum of quadratic parts of hamiltonian about $X = J$ and $X = 0$ vacua, assuming that the cubic and higher order terms are small. To

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verify this assumption we need to keep the cubic and higher order terms in the expansion and obtain the coefficient in front of these terms, the coupling of the theory. However, first we should bring the quadratic parts of the hamiltonian to the "canonical" form

$$H^{(2)} \sim \sum \omega_j \mathcal{A}_j^\dagger \mathcal{A}_j \, ,$$

where $\mathcal{A}_j$'s are generic creation operators satisfying $[\mathcal{A}_j, \mathcal{A}_{j'}^\dagger] = \delta_{jj'}$ (note that this commutator is a quantum operator commutator and not a matrix commutator). This can be done by rescaling X's as

$$X^i \to \sqrt{\frac{R_-}{\mu}} X^i \, , \qquad X^a \to \sqrt{\frac{R_-}{\mu}} X^a \, , \tag{3.20}$$

(no scaling for fermions is needed). One can then observe that the coefficient in front of order m-terms ($m = 3, \ldots, 6$) is $\left(\frac{R_-}{\mu\sqrt{g_s}}\right)^{m-2}$. Therefore, the "bare" coupling of the theory is:

$$g^2_{\text{bare}} = \frac{R_-^2}{\mu^2 g_s} \, . \tag{3.21}$$

The reason we have called the above coupling, the bare coupling is that generically there are dressing factors (power of J) which are multiplied to g^2_{bare}. The effective coupling and the dressing factors of course depend about which vacuum we are expanding the theory (for a similar discussion on BMN Matrix model see [6]). As we see from (3.21), we can trust our perturbation theory analysis when $g_{\text{bare}} \ll 1$. That is we are working in the large μ limit and performing a $1/\mu$ expansion.

Effective coupling about $X = 0$ vacuum: here we deal with a U(J) gauge theory. Despite the fact our gauge theory is not a Yang-Mills theory, we expect the 't Hooft arguments about the planar and non-planar expansion to still hold because, 't Hooft's arguments are based on combinatorics of $J \times J$ matrices and not details of the model. Therefore, we expect the effective coupling of the theory about this vacuum to be the 't Hooft coupling:

$$g_{X=0} = g^2_{\text{bare}} J = \frac{J^3}{(\mu p^+)^2 g_s} \, . \tag{3.22}$$

The above should, however, be verified through explicit calculations with our model, e.g. by computing corrections to the spectrum of our states, employing time-independent quantum mechanical perturbation theory. Then, $g_{X=0}$, and not g_{bare}, should appear in the mass corrections.

In the string theory limit (2.13), the effective coupling $g_{X=0}$ blows up ($\sim J^3$) and hence in this limit the $X = 0$ vacuum is quite non-trivial and strongly coupled and one cannot trust perturbative information, expect for the BPS (protected) data.

Effective coupling about $X = J$ vacuum: working out the dressing factors here are tricker than the previous case and we need to develop an extra knowledge about Spin(4) representations and its Clebsch-Gordon coefficients. However, noting that Spin(4) =

$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ paves the way to use the more standard information of SU(2) Clebsch-Gordon coefficients and in particular Wigner $3j$ and $6j$ symbols (e.g. see [25]). A similar analysis has been done in [6]. For large J, and hence large n, where n is the highest spin of either of the $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$, (recall that $J \sim n^2$ (D.11)), these $6j$-symbols lead to a factor of $n^{-3/2}$. However, one should remember that we have two SU(2)'s and there is a factor of $1/\sqrt{n}$ coming from the normalization factors of the other SU(2). So, altogether we expect to have a dressing factor of $\frac{1}{n^2} \sim \frac{1}{J}$ and consequently

$$g_{X=J} = g_{\rm bare}\frac{1}{J} = \frac{R_-}{\mu\sqrt{g_s}} = \frac{1}{\mu p^+\sqrt{g_s}}\,. \tag{3.23}$$

This is compatible with the results of [16]. It is interesting to note that $g^2_{X=J}$ is indeed inverse of the effective coupling of strings on the plane-wave backgroundg_2 (in terms of BMN gauge theory parameters $g_2 = J^2/N$) [16]. In the string theory limit $g_{X=J}$ remains finite and hence the giant graviton theory is a perturbatively accessible theory.

4. Conclusions and discussion

In this paper we conjectured that type IIB string theory on the plane-wave background, in the DLCQ description, is governed by a $0+1$ dimensional gauge theory, the tiny graviton Matrix theory, with the hamiltonian (2.11). In this theory, which is the holographic description of strings on the plane-wave, fundamental objects are the tiny gravitons, which are spherical 3-branes of very small size. We gave preliminary set of evidence in support of this conjecture, analyzed vacua of the theory and discussed a continuum limit in which we recover the ten dimensional string theory.

There is another description of the same theory, the BMN gauge theory, in which, unlike our case, fundamental closed strings are perturbative. Following the procedure of taking the Penrose limit and recalling the analysis of BMN more carefully we observe that in terms of $\mathcal{N}=4, D=4, \mathrm{U}(N)$ SYM parameters $R_-/\mu = \sqrt{g_s N}$ and hence the fuzziness parameter (3.4), in the ten dimensional Planck units, is

$$l^2 = \sqrt{\frac{1}{N}}l^2_{\rm Pl} = R^2_{\rm tiny}\,. \tag{4.1}$$

For the second equality we have used (2.3) and (2.4). This result is very interesting: the same parameter which controls the fuzziness, l, which is also the size of tiny gravitons, is exactly the same as the parameter governing string interactions, $1/N$. The above together with (2.13) suggests that the smooth string picture in the plane-wave or $AdS_5 \times S^5$ background only emerges in the $N \to \infty$ limit.

Here we also briefly comment on the possible connection between our model about $X = 0$ vacuum and the string bit model [23], and also a way which would hopefully help to reconstruct the tension term in the light-cone string σ-model action. In string bit model strings are composed of J bits, instead of $J \times J$ matrices. The string bit model is a supersymmetric quantum mechanics of J particles with $\mathrm{PSU}(2|2) \times \mathrm{PSU}(2|2) \times \mathrm{U}(1)$ superalgebra built in [23]. A possible connection to our Matrix model may come from identifying the string bit coordinate with the diagonal elements of our matrices. It is easy to see that a diagonal matrix of the form

$$X^I = \begin{pmatrix} x_1^I & & & & & \\ & x_2^I & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & x_J^I \end{pmatrix}, \tag{4.2}$$

can be a classical solution. In addition, (4.2) is not a gauge invariant solution of our Matrix model. (Under a generic $\mathrm{U}(J)$ gauge tranformation it loses its diagonal form.) However, if we restrict our gauge transformations to $\mathcal{S}_J \subset \mathrm{U}(J)$ ($\mathcal{S}_J$ is the group of permutation of J objects), the diagonal matrices remain diagonal. Interestingly, $\mathcal{S}_J$ is the symmetry group of the string bit model [23]. Hence one may hope that it should be possible, to recover string bit model once we make the restriction to $\mathcal{S}_J$ gauge transformation, or to argue that quantum mechanically $\mathrm{U}(J)$ symmetry is naturally reduces to $\mathcal{S}_J$.

Moreover, presumably the quadratic parts of the string bit model hamiltonian would then appear as the one loop effective action in our Matrix model, once we integrate out next to diagonal elements of matrices about the strong coupling fixed point. In other words, the string tension terms, which in the discrete format takes the form $\sum \frac{1}{J}(x_{j+1}^I - x_j^I)^2$, stems from the interactions between the tiny gravitons. (Note also that in our quantization J^{-1} play the role of $\hbar$ and as the tension term comes with a factor of $1/J$ it is reasonable to think of it as a term in the one loop effective action.)

It has been shown that the plane-wave with a compact X^- naturally appears as a certain Penrose limit of $AdS_5 \times S^5/Z_k$ orbifold in the large k limit [26]. As the $AdS_5 \times S^5/Z_k$ is dual to an $\mathcal{N} = 2$ $\mathrm{SU}(N)^k$ super-conformal gauge theory, this implies that DLCQ of strings on the plane-wave has a natural description in the "BMN sector" of this gauge theory. It would be nice to make connection between our formulation and the BMN sector of the $\mathcal{N} = 2$ $D = 4$ super-conformal gauge theory .

The other interesting question is that whether our proposal is good enough to provide us with a $0 + 1$ dimensional gauge theory description for DLCQ of type IIB strings in the flat space. At the level of the hamiltonian (2.11), flat space limit, i.e. $\mu \to 0$ limit, seems to be a smooth one. However, as we have shown in section 3.3, all the vacua of the Matrix model become strongly coupled and our perturbative $1/\mu$ expansion breaks down. Therefore, although this limit exists, the theory becomes intractable. It would also be interesting to understand how our Matrix model at $\mu = 0$ and the other type IIB string Matrix theories, the IKKT [27] and Susskind-Sethi [28] model, are related to each other. (The former is a quantum mechanics of instantons $0 + 0$ gauge theory and the latter is a $2 + 1$ SYM.)

Finally we would like to comment on whether our method for quantization of of Nambu odd brackets can be used for quantizing a six dimensional $(0, 2)$ theory, the theory which lives on spherical M5-brane giants. If our method is applicable, similarly to the 3-brane case, the quantized M5-brane theory should become a theory on a fuzzy five sphere. This idea, however, does not pass the first test: the relation between the radius of a S_F^5 and

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the size of matrices describing it is $(R/l)^5 \sim J$ (cf. (D.4)), whereas for a five sphere giant $(R/l)^4 \sim \mu p^+$ [7, 19]. Of course this argument does not completely rule out the applicability of this idea and this direction deserves further analysis.

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A. Light-cone hamiltonian of a 3-brane in the plane-wave background

As we discussed our tiny graviton matrix theory conjecture is based on the light-cone quantization (regularization) of a 3-brane action in the maximally supersymmetric plane-wave background. In this appendix we review, essentially quoting from [16], how to obtain the light-cone hamiltonian. We start with the Born-Infeld action for a 3-brane:

$$S = \frac{1}{l_s^4 g_s} \int d\tau d^3\sigma \sqrt{- \det \left(G_{\mu\nu} \partial_{\hat{\mu}} X^\mu \partial_{\hat{\nu}} X^\nu\right)} + \int C_4 \,, \tag{A.1}$$

where $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$ indices correspond to the worldvolume coordinates τ, σ^r, $r = 1, 2, 3$. X^μ are the embedding coordinates and $G_{\mu\nu}$ and C_4 are the background plane-wave metric and fourforms (1.1) and (1.2). Note that we have *not* included the U(1) gauge field of the brane. In the whole paper we set $l_s = 1$ and unless it is emphasized explicitly, all our quantities are measured in units of l_s.

To fix the light-cone gauge we set

$$X^+ = \tau \tag{A.2a}$$

$$g_{0r} \equiv G_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma^r} = 0 \,. \tag{A.2b}$$

The momentum conjugate to X^+, $\frac{\partial \mathcal{L}}{\partial \partial_\tau X^+}$, is the light-cone hamiltonian H_{lc} and the momentum conjugate to X^-, $\frac{\partial \mathcal{L}}{\partial \partial_\tau X^-}$, is the light-cone momentum p^+. The momentum conjugate to the directions transverse to the light-cone, X^I, $I = 1, 2, \ldots, 8$ is $P_I = \frac{\partial \mathcal{L}}{\partial \partial_\tau X^I}$.

Fixing the light-cone gauge fixes a part of the diffeomorphism invariance of the Born-Infeld action which mixes the worldvolume time, τ, with the worldvolume spatial directions σ^r. The part of the diffeomorphisms which only act on the spatial directions, however, are still present and not fixed. It is noteworthy that as always in the light-cone gauge fixing there is no need of introducing ghosts (the ghosts are decoupled).

Simplifying the light-cone hamiltonian, we obtain [16]

$$\begin{aligned} H^{\text{bos.}}_{\text{l.c.}} = \int d^3\sigma \bigg[& \frac{1}{2p^+}(P_i^2 + P_a^2) + \frac{\mu^2 p^+}{2}(X_i^2 + X_a^2) + \frac{1}{2p^+ g_s^2} \det g_{rs} - \\ & - \frac{\mu}{6g_s} \left(\epsilon^{ijkl} X^i \{X^j, X^k, X^l\} + \epsilon^{abcd} X^a \{X^b, X^c, X^d\} \right) \bigg] \,, \end{aligned} \tag{A.3}$$

where

$$g_{rs} = \partial_r X^i \partial_s X^i + \partial_r X^a \partial_s X^a \,,$$

and the brackets are Nambu three brackets [31] and are defined as

$$\{F, G, K\} = \epsilon^{prs} \partial_p F \partial_r G \partial_s K \,, \tag{A.4}$$

where F, G and K are arbitrary functions of the worldvolume coordinates σ^r. The $\det g_{rs}$ term can also be written in terms of Nambu brackets [30]

$$\begin{aligned} \det g_{rs} = & \frac{1}{3!}\{X^i, X^j, X^k\}\{X^i, X^j, X^k\} + \frac{1}{3!}\{X^a, X^b, X^c\}\{X^a, X^b, X^c\} + \\ & + \frac{1}{2!}\{X^i, X^j, X^a\}\{X^i, X^j, X^a\} + \frac{1}{2!}\{X^i, X^a, X^b\}\{X^i, X^a, X^b\} \,. \end{aligned} \tag{A.5}$$

The first line of (A.3) is coming from the Born-Infeld term and the second line from the Chern-Simons term (C_4 term) of (A.1). The term proportional to μ^2 comes from the $(dx^+)^2$ term of the metric (1.1) and the terms linear in μ from the fiveform of the background.

The above action, besides the diffeomorphism on σ^r directions (which is a local gauge symmetry), has an SO(4) $\times$ SO(4) global symmetry, the SO(4)'s acting on X^i or X^a directions, respectively. Moreover, there is a Z_2 symmetry which exchanges X^i and X^a directions [16]. The condition (A.2b),

$$p^+ \partial_r X^- \approx P_i \partial_r X^i + P_a \partial_r X^a \,, \tag{A.6}$$

where $\approx$ is the "weak" equality, should be imposed as a constraint on the dynamics of the system (or on the states, in the quantized version). Hence the dynamics of X^- in the light-cone gauge is completely determined through dynamics of the transverse directions.

The hamiltonian (A.3) is the bosonic part of the full supersymmetric hamiltonian with superalgebra PSU(2|2) $\times$ PSU(2|2) $\times$ U(1) [16, 8]. The fermionic part of the hamiltonian after fixing the κ symmetry in the light-cone gauge is [29, 16]

$$\begin{aligned} H^{\rm fer}_{\rm l.c.} = \int d^3\sigma \Big[& \mu \ \psi^{\dagger \alpha\beta} \psi_{\alpha\beta} + \frac{2}{p^+ g_s} \left(\psi^{\dagger \alpha\beta} (\sigma^{ij})_\alpha{}^\delta \{X^i, X^j, \psi_{\delta\beta}\} + \psi^{\dagger \alpha\beta} (\sigma^{ab})_\alpha{}^\delta \{X^a, X^b, \psi_{\delta\beta}\} \right) - \\ & - \mu \ \psi_{\dot\alpha\dot\beta} \psi^{\dagger \dot\alpha\dot\beta} - \frac{2}{p^+ g_s} \left(\psi_{\dot\delta\dot\beta} (\sigma^{ij})_{\dot\alpha}{}^{\dot\delta} \{X^i, X^j, \psi^{\dagger \dot\alpha\dot\beta}\} + \psi_{\dot\delta\dot\beta} (\sigma^{ab})_{\dot\alpha}{}^{\dot\delta} \{X^a, X^b, \psi^{\dagger \dot\alpha\dot\beta}\} \right) \Big] \,. \end{aligned} \tag{A.7}$$

Following the notation of [8], the fermions ψ are spinors of two different SU(2)$'s$, one coming from the decomposition of each of the two SO(4)'s into SU(2) $\times$ SU(2); in other words, ψ carries two spinor indices, each being the Weyl index of one of the SO(4)'s. The fact that in type IIB both of fermions have the same ten dimensional chirality is reflected in the fact that both of our fermions should have the same SO(4) chiralities [8]. The explicit mass terms (terms proportional to μ) in $H^{fer}_{l.c.}$ for the fermions come from the contribution of RR fiveforms to the super-vielbeins.

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B. Nambu brackets and their quantization

As we showed in appendix A, Nambu 3-brackets naturally appear in the light-cone hamiltonian of a 3-brane. In fact, following the logic of previous appendix, it is straightforward to show that in general a Nambu p-bracket appears in the p-brane light-cone hamiltonian; for the case of M2 and M5 -branes this has been explicitly shown [6, 19, 5]. In this appendix, first we define a general Nambu p-bracket and list its properties and then discuss quantization of brackets.

B.1 Properties of Nambu brackets

In general a Nambu p-bracket [31] is defined among p functions $F_i(\sigma^r)$, $i = 1, 2, \ldots, p$, where F_i are (real) functions on a p-dimensional space, parametrized with σ^r, $r = 1, 2, \ldots, p$:

$$\{F_1, F_2, \ldots, F_p\} \equiv \epsilon^{r_1 r_2 \cdots r_p} \frac{\partial F_1}{\partial \sigma^{r_1}} \frac{\partial F_2}{\partial \sigma^{r_2}} \cdots \frac{\partial F_p}{\partial \sigma^{r_p}} . \tag{B.1}$$

For $p = 2$ the above reduces to the usual Poisson bracket. The above bracket has five important properties:

i) Cyclicity

$$\{F_1, F_2, \ldots, F_i, \ldots, F_j, \ldots, F_p\} = -\{F_1, F_2, \ldots, F_j, \ldots, F_i, \ldots, F_p\} . \tag{B.2}$$

Consequently, under the cyclic rotation of functions inside bracket:

$$\{F_1, F_2, \ldots, F_p\} = (-1)^{p+1}\{F_2, F_3, \ldots, F_p, F_1\} . \tag{B.3}$$

Eq. (B.3) is a consequence of a similar property of the ϵ symbol, namely

$$\epsilon^{r_1 r_2 \cdots r_p} = (-1)^{p+1} \epsilon^{r_2 \cdots r_p r_1} . \tag{B.4}$$

Therefore, even and odd Nambu brackets have a different cyclicity behaviour. As we will see in the next subsection this difference is the source of difficulties with quantization of Nambu odd brackets.

ii) Jacobi Identity

$$\epsilon^{i_1 i_2 \cdots i_{2p-1}} \Big\{F_1, F_2, \ldots, F_{p-1}, \{F_p, F_{p+1}, \ldots, F_{2p-1}\}\Big\} = 0 . \tag{B.5}$$

iii) Associativity

$$\{F_1, F_2, \ldots, F_{p-1}, F_p G_p\} = \{F_1, F_2, \ldots, F_{p-1}, F_p\} G_p + F_p \{F_1, F_2, \ldots, F_{p-1}, G_p\} . \tag{B.6}$$

iv) Trace property

$$\int d^p\sigma \{F_1, F_2, \ldots, F_{p-1}, F_p\} = 0 . \tag{B.7}$$

v) By part integration

$$\int d^p\sigma \{F_1, F_2, \ldots, F_{p-1}, F_p\} G_p = -\int d^p\sigma \{F_1, F_2, \ldots, F_{p-1}, G_p\} F_p . \tag{B.8}$$

All the above properties can be verified directly from the definition (B.1). Note also that the integration by part property is a result of iii) and iv).

B.2 Quantization of Nambu brackets

The formulation of standard quantum mechanics is obtained from the usual Hamilton-Jacobi formulation of classical mechanics once we replace the Poisson brackets (i.e. Nambu two brackets) with their "quantized" version, the matrix (or operator) commutators:

$$\{A, B\} \longleftrightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}]\,. \tag{B.9}$$

Since Nambu introduced his brackets [31] their quantization has been extensively studied and in particular quantization of Nambu odd brackets has appeared to be very challenging e.g. see [32].

B.2.1 Quantization of Nambu even brackets

Prescription:

i) Replace functions $F_i(\sigma)$ with operators (matrices) $\hat{F}_i$.

ii)

$$\{F_1, F_2, \ldots, F_{2p}\} \longleftrightarrow \frac{1}{i^p}[\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_{2p}] \equiv \frac{1}{i^p}\frac{1}{(2p)!}\epsilon^{i_1 i_2 \cdots i_{2p}} \hat{F}_{i_1}\hat{F}_{i_2}\cdots\hat{F}_{i_{2p}}\,. \tag{B.10}$$

iii)

$$\int d^{2p}\sigma \, * \longleftrightarrow \mathrm{Tr}\, *\,. \tag{B.11}$$

As wee see from (B.10) the $2p$-bracket can be written in terms of a sum of products of p commutators. Note also that, similarly to $\hbar$ in (B.9), generically we need to introduce a "quantization" parameter in (B.10).

Using definition (B.10) one can easily check that the Jacobi identity (B.5) is satisfied, however, for $p > 1$ the associativity is compromised [32]. The trace property (B.7), follows from (B.4), the cyclicity of the trace and eq. (B.11). The by part integration property, despite the absence of associativity, is still satisfied.

B.2.2 Quantization of Nambu odd brackets

Noting (B.4), it is easy to observe that if we use prescription (B.10) for odd brackets we lose the trace and by part integration properties. These two properties are physically very important because they are connected with the (existence of) conserved charges. Therefore, (B.10) cannot be directly used for the odd brackets.

Here we propose a solution for the problem of quantization of Nambu odd brackets, which as we will see in the main text and also in appendix D, is a suitable one for the cases of our concern, namely the cases where the p brackets come form the worldvolume of the p dimensional branes. The idea is to replace $2p-1$ brackets with $2p$ brackets and again use (B.10). Explicitly:

$$\{F_1, F_2, \ldots, F_{2p-1}\} \longleftrightarrow \frac{1}{i^p}[\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_{2p-1}, \hat{\mathcal{L}}_{2p+1}] \equiv \frac{1}{i^p(2p)!}\epsilon^{i_1 i_2 \cdots i_{2p}} \hat{F}_{i_1}\hat{F}_{i_2}\cdots\hat{F}_{i_{2p-1}}\hat{\mathcal{L}}_{2p+1}\,, \tag{B.12}$$

where $\hat{\mathcal{L}}_{2p+1}$ is a given fixed matrix. This matrix is closely related to the "chirality" operator for $2p$ dimensional fermions. In the case of our interest, namely, Nambu 3-brackets where the explicit expression for (B.12) is

$$\begin{aligned}[A,B,C,\mathcal{L}_5] = \frac{1}{24}&([A,B][C,\mathcal{L}_5] - [A,C][B,\mathcal{L}_5] + [A,\mathcal{L}_5][B,C] + \\ &+ [C,\mathcal{L}_5][A,B] - [B,\mathcal{L}_5][A,C] + [B,C][A,\mathcal{L}_5])\,,\end{aligned} \tag{B.13}$$

and $\mathcal{L}_5$ is made out of direct product of $\mathbf{1}_{4\times4}$ and Γ^5, where Γ^5 is the SO(4) chirality operator. This point will be discussed in more detail in appendix D.

C. Superalgebra of the plane-wave background

The plane-wave solution (1.1) has a large set of bosonic and fermionic isometries. The bosonic isometry group, whose dimension is 30, contains SO(4) $\times$ SO(4) and translation along x^- and x^+ directions, the generators of which will be denoted by $\mathbf{J_{ij}}$, $\mathbf{J_{ab}}$, P^+ and $\mathbf{H}$, respectively [8].

There are also 32 fermionic isometries (supercharges) which can be decomposed into 16 kinematical supercharges $q_{\alpha\beta}$, $q_{\dot\alpha\dot\beta}$ (and their complex conjugates) and 16 dynamical supercharges $Q_{\alpha\dot\beta}$ and $Q_{\dot\alpha\beta}$. $P^+ = i\frac{\partial}{\partial x^-}$ is at the center of the whole superalgebra, i.e. it commutes with all the bosonic and fermionic generators. The dynamical part of the superalgebra is (for the full superalgebra see [8]):

$$\begin{aligned}
\{q_{\alpha\beta}, q^{\dagger\rho\lambda}\} &= \delta^\rho_\alpha\ \delta^\lambda_\beta\ P^+\,, \qquad \{q_{\dot\alpha\dot\beta}, q^{\dagger\dot\rho\dot\lambda}\} = \delta^{\dot\rho}_{\dot\alpha}\ \delta^{\dot\lambda}_{\dot\beta}\ P^+\,,\\
[P^+, q] &= 0\,, \qquad [P^+, q^\dagger] = 0\\
[\mathbf{H}, q_{\alpha\beta}] &= \mu q_{\alpha\beta}\,, \qquad [\mathbf{H}, q_{\dot\alpha\dot\beta}] = -\mu q_{\dot\alpha\dot\beta}\,.
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
[P^+, Q] &= 0\,, \qquad [P^+, Q^\dagger] = 0\\
[\mathbf{H}, Q] &= 0\,, \qquad [\mathbf{H}, Q^\dagger] = 0\,.
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\{Q_{\alpha\dot\beta}, Q^{\dagger\rho\dot\lambda}\} &= \delta^\rho_\alpha\ \delta^{\dot\lambda}_{\dot\beta}\ \mathbf{H} + 2\mu(i\sigma^{ij})^\rho_\alpha\delta^{\dot\lambda}_{\dot\beta}\ \mathbf{J}_{ij} + 2\mu(i\sigma^{ab})^{\dot\lambda}_{\dot\beta}\delta^\rho_\alpha\ \mathbf{J}_{ab}\\
\{Q_{\dot\alpha\beta}, Q^{\dagger\dot\rho\lambda}\} &= \delta^{\dot\rho}_{\dot\alpha}\ \delta^\lambda_\beta\ \mathbf{H} + 2\mu(i\sigma^{ij})^{\dot\rho}_{\dot\alpha}\delta^\lambda_\beta\ \mathbf{J}_{ij} + 2\mu(i\sigma^{ab})^\lambda_\beta\delta^{\dot\rho}_{\dot\alpha}\ \mathbf{J}_{ab}\,.
\end{aligned} \tag{C.3}$$

The above superalgebra (C.2) and (C.3) (involving Q's, $\mathbf{H}$ and P^+) can be identified as PSU(2|2) $\times$ PSU(2|2) $\times$ U(1)$_-$ $\times$ U(1)$_+$ where the U(1)$_\pm$ are translation along the $x^\pm$ directions. The bosonic part of PSU(2|2) algebra is SU(2) $\times$ SU(2) where each of SU(2)'s is coming from a different SO(4) isometry of the background.

C.1 SUSY algebra in terms of $J \times J$ matrices

The generators of the above supersymmetry algebra can be realized in terms of $J \times J$ matrices as

$$P^+ = \frac{1}{R_-}\mathrm{Tr}\mathbf{1}\,, \tag{C.4a}$$

$$q_{\alpha\beta} = \frac{1}{\sqrt{R_-}}\ \mathrm{Tr}\psi_{\alpha\beta}\,, \qquad q_{\dot\alpha\dot\beta} = \frac{1}{\sqrt{R_-}}\ \mathrm{Tr}\psi_{\dot\alpha\dot\beta}\,. \tag{C.4b}$$

$$\mathbf{J}_{ij} = \mathrm{Tr}\left(X^i\Pi^j - X^j\Pi^i + \psi^{\dagger\alpha\beta}(i\sigma^{ij})_\alpha^{\ \rho}\psi_{\rho\beta} - \psi^{\dagger\dot\alpha\dot\beta}(i\sigma^{ij})_{\dot\alpha}^{\ \dot\rho}\psi_{\dot\rho\dot\beta}\right) \tag{C.5a}$$

$$\mathbf{J}_{ab} = \mathrm{Tr}\left(X^a\Pi^b - X^b\Pi^a + \psi^{\dagger\alpha\beta}(i\sigma^{ab})_\beta^{\ \rho}\psi_{\alpha\rho} - \psi^{\dagger\dot\alpha\dot\beta}(i\sigma^{ab})_{\dot\beta}^{\ \dot\rho}\psi_{\dot\alpha\dot\rho}\right) \tag{C.5b}$$

$$\begin{aligned} Q_{\dot\alpha\beta} = \sqrt{\frac{R_-}{2}}\ \mathrm{Tr}\Bigg[&\left(\Pi^i - i\frac{\mu}{R_-}X^i\right)(\sigma^i)_{\dot\alpha}^{\ \rho}\psi_{\rho\beta} + \left(\Pi^a - i\frac{\mu}{R_-}X^a\right)(\sigma^a)_\beta^{\ \dot\rho}\psi_{\dot\alpha\dot\rho} + \\ &+\frac{1}{3!g_s}\left(\epsilon^{ijkl}[X^i,X^j,X^k,\mathcal{L}_5](\sigma^l)_{\dot\alpha}^{\ \rho}\psi_{\rho\beta} + \epsilon^{abcd}[X^a,X^b,X^c,\mathcal{L}_5](\sigma^d)_\beta^{\ \dot\rho}\psi_{\dot\alpha\dot\rho}\right) + \\ &+\frac{1}{2g_s}\Big([X^i,X^a,X^b,\mathcal{L}_5](\sigma^i)_{\dot\alpha}^{\ \rho}(i\sigma^{ab})_\beta^{\ \gamma}\psi_{\rho\gamma} + \\ &\qquad + [X^i,X^j,X^a,\mathcal{L}_5](i\sigma^{ij})_{\dot\alpha}^{\ \dot\rho}(\sigma^a)_\beta^{\ \dot\gamma}\psi_{\dot\rho\dot\gamma}\Big)\Bigg] \end{aligned} \tag{C.6}$$

and similarly for $Q_{\alpha\dot\beta}$. The expression for $\mathbf{H}$ is given in (2.11).

The (anti)commutation relations (C.1), (C.2) and (C.3) may be verified using the quantum (as opposed to matrix) commutation relations:

$$[X^I_{pq}, \Pi^J_{rs}] = i\delta^{IJ}\ \delta_{ps}\delta_{qr}\,, \qquad I,J = 1,2,\ldots,8 \tag{C.7a}$$

$$\{(\psi^{\dagger\alpha\beta})_{pq}, (\psi_{\rho\gamma})_{rs}\} = \delta^\alpha_\rho\delta^\beta_\gamma\ \delta_{ps}\delta_{qr}\,, \qquad \{(\psi^{\dagger\dot\alpha\dot\beta})_{pq}, (\psi_{\rho\gamma})_{rs}\} = \delta^{\dot\alpha}_{\dot\rho}\delta^{\dot\beta}_{\dot\gamma}\ \delta_{ps}\delta_{qr}\,, \tag{C.7b}$$

where $p,q,r,s = 1,2,\ldots,J$ are matrix indices.

As for the kinematical supersymmetry generators, here we have only presented q's. The bosonic kinematical isometries can be worked out in a similar manner, through anti-commutation of a dynamical supercharge and a kinematical one [8]. The kinematical super-generators, are all in the U(1) (trace) part of the U(J) matrices.

D. A brief review on fuzzy d spheres, S^d_F

Fuzzy spheres are (noncommutative) discretized spheres and the discretization (" quantization") of the sphere is performed in such a way that the SO($d+1$) rotation symmetry of a d sphere, S^d, is preserved. In order to give an idea how this quantization can be done, recall that a round d sphere of radius R can be embedded in a $d+1$ dimensional flat space as

$$\sum_{i=1}^{d+1}(X^i)^2 = R^2\,. \tag{D.1}$$

Next note that for any d sphere the coordinates satisfy

$$\{X^{i_1}, X^{i_2}, \ldots, X^{i_d}\} = R^{d-1}\epsilon^{i_1 i_2\ldots i_{d+1}}X^{i_{d+1}}\,, \tag{D.2}$$

where the left-hand-side is a Nambu d bracket.

To see this more clearly let us consider the simple example of a two sphere, in which case the above bracket is essentially a Poisson bracket. Take the following embedding:

$$X^1 = R\sin\theta\cos\phi\,, \qquad X^2 = R\sin\theta\sin\phi\,, \qquad X^3 = R\cos\theta\,,$$

where $\theta,\ \phi$ are the spherical coordinates. It is then straightforward to check that

$$\{X^1, X^2\} = \frac{1}{\sin\theta}\left(\partial_\theta X^1 \partial_\phi X^2 - \partial_\theta X^2 \partial_\phi X^1\right) = R^2\cos\theta = RX^3\,,$$

(Note that $1/\sin\theta$ factor is the volume-form on the S^2) and in general $\{X^i, X^j\} = R\epsilon^{ijk}X^k$. This point for $d = 3, 5$ has been noted and used in [16, 19]. The advantage of using (D.1) and (D.2) is that they are both invariant under the $\mathrm{SO}(d+1)$ rotations.

D.1 Fuzzifying an S^d

To quantize or "fuzzify" the d sphere, we quantize the Nambu bracket (D.2) using the prescription of quantizing Nambu brackets discussed in appendix B. The equation (D.1) is then simply a condition on the size of the matrices. In other words, we replace X^i's with the matrices which are finite dimensional representations of $\mathrm{SO}(d+1)$ and (D.1), or the radius, is the second rank Casimir for that representation. In the process of quantization of Nambu brackets we need to introduce a parameter, l, which measures the amount of fuzziness. This parameter plays the role of $\hbar$ in the usual quantum mechanics and is defined in such a way that in the $l \to 0$ limit (while the radius is kept fixed) we recover the commutative round sphere. To show how the quantization works and as a warm up we consider the case of fuzzy two sphere S^2_F. In this case the quantum version of (D.2) is

$$[X^i, X^j] = il\epsilon^{ijk}X^k\,,$$

where l is the fuzziness and X^i/l are generators of $\mathrm{SO}(3) \simeq \mathrm{SU}(2)$. The S^2_F of radius R is then given in terms of a $J \times J$ representation of $\mathrm{SU}(2)$, where (e.g. see [6])

$$\left(\frac{R}{l}\right)^2 = \frac{J^2-1}{4}\,. \tag{D.3}$$

In order to fuzzify an even sphere we can simply generalize the above, using the arguments of appendix B. The case of fuzzy odd spheres, in which we face odd Nambu brackets, is trickier. A nice and detailed discussion on fuzzy spheres, even and odd, may be found in [33, 34]. (We would like to note that the above method for quantizing or fuzzifying a sphere, which is quite generic and works for even and odd cases alike, is a new construction and to the author's knowledge it has not been presented in the literature previously. This method is built in such a way that in the continuum limit, $J \to \infty$, $R = fixed$, we recover the round commutative sphere.)

Before moving to a more detailed discussion on the S^3_F case, we quote an important result about the fuzzy spheres. In general for a S^d_F $(d \geq 2)$ the size of the matrices, $J \times J$, and the radius R, for large J, are related as [34]

$$\left(\frac{R}{l}\right)^{\frac{1}{8}d(d+2)} \sim J \qquad \text{for even } d\,, \tag{D.4a}$$

$$\left(\frac{R}{l}\right)^{\frac{1}{8}(d-1)(d+5)} \sim J \qquad \text{for odd } d\,. \tag{D.4b}$$

For $d = 2$ $(R/l) \sim J$ (reducing to (D.3)), for $d = 3$ $(R/l)^2 \sim J$ and $(R/l)^3 \sim J$ for $d = 4$.

D.2 Specific case of fuzzy three sphere S_F^3

Now we focus on the case relevant to our tiny graviton Matrix model, the fuzzy three sphere, S_F^3. The S_F^3 is described through the embedding matrices X^i which satisfy[8]

$$[X^i, X^j, X^k, \mathcal{L}_5] = -l^2 \epsilon^{ijkl} X^l \,, \tag{D.5a}$$

$$\sum_{i=1}^{4} (X^i)^2 = R^2 \,. \tag{D.5b}$$

To give an idea how to solve (D.5a), let us start with the 4×4 matrices, i.e.

$$X^i = l\Gamma^i \,, \qquad \mathcal{L}_5 = \Gamma^5 \,, \tag{D.6}$$

where Γ's are the standard four dimensional Dirac matrices. It is evident that (D.6) solves (D.5) for $R = 2l$. In other words (D.6) defines a S_F^3 of radius two. The necessity of the existence of $\mathcal{L}_5$ in the definition of (B.12) or (D.5) is clearly visible, if we demand to have a solution in the form of Γ matrices.[9]

To construct a fuzzy three sphere of generic radius, we start with matrices made out of a sequence of n Γ's and 4×4 identity matrices [33, 34] :

$$L^i \equiv (\Gamma^i \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma^i \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \Gamma^i)_{\mathrm{sym}} \,, \tag{D.7a}$$

$$L_5 \equiv (\Gamma^5 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma^5 \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \Gamma^5)_{\mathrm{sym}} \,, \tag{D.7b}$$

The above would guarantee that the L_i and L_5 solve (D.5a). L_i's, however, are not defining an S_F^3 of definite radius. To obtain a fuzzy three sphere of definite radius, we need to restrict L_i's to specific (reducible) representations of $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. If we call such a representation $\mathcal{R}$, i.e.

$$\frac{X^i}{l} = P_{\mathcal{R}} L_i P_{\mathcal{R}} \,, \qquad \mathcal{L}_5 = P_{\mathcal{R}} L_5 P_{\mathcal{R}} \,, \tag{D.8}$$

where $P_{\mathcal{R}}$ is the projector restricting L's to $\mathcal{R}$. Size of the matrices J which is defined in (D.8) is[10]

$$J = \dim \mathcal{R} \,. \tag{D.9}$$

The requirement that $\sum (X^i)^2 = R^2$ is proportional to identity matrix, utilizing Schur's Lemma, implies that $\mathcal{R}$ is a sum of irreducible representations of SO(4) moreover, these representations should have the same $\sum (L^i)^2$ eigenvalues. This implies that [33]

$$\mathcal{R} = \left(\frac{k}{2}, \frac{n-k}{2}\right) \oplus \left(\frac{n-k}{2}, \frac{k}{2}\right),$$

[8] Another construction for S_F^3/Z_2 has been presented in [35].

[9] Had we taken $X^i = l\Gamma^i\Gamma^5$,

$$[X^i, X^j, X^k, \mathcal{L}_5] = +l^2 \epsilon^{ijkl} X^l \,.$$

Based on the discussions of section 3, this choice corresponds to an anti fuzzy sphere, i.e. a S_F^3 with orientation opposite to that of $X^i = l\Gamma^i$ solution. Anti-giants are solutions of hamiltonian (2.11) with $\mu \to -\mu$.

[10] We would like to stress that the $\mathcal{L}_5$ which is used in the Hamiltonian (2.11) is only subject to (D.8) and (D.9); eqs. (D.10) and (D.11) only give a specific solution.

where (j_L, j_R) denotes an irreducible representation of $\mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and j_L and j_R are integer or half-integer valued. In addition, in order X^i to be non-trivial in this representation, L_i must be a map between the two irreducible components of $\mathcal{R}$, which implies that $k = n - k \pm 1$. This relation restricts n to be an *odd* integer and also

$$\mathcal{R} = \left(\frac{n-1}{4}, \frac{n+1}{4}\right) \oplus \left(\frac{n+1}{4}, \frac{n-1}{4}\right). \tag{D.10}$$

Dimension of representation $\mathcal{R}$, i.e. size of matrices J, is

$$J = 2 \cdot \left[2\left(\frac{n+1}{4}\right) + 1\right]\left[2\left(\frac{n-1}{4}\right) + 1\right] = \frac{1}{2}(n+1)(n+3)\,, \qquad n = \text{odd}\,. \tag{D.11}$$

Therefore, the allowed values for size of the matrices, unlike the S^2_F case, cannot take any arbitrary integer values. In particular the lowest J's are $4, 6, 12, \ldots$.

Using the above one can show that [34]

$$\begin{aligned}\sum \left(\frac{X^i}{l}\right)^2 &= \sum (P_{\mathcal{R}} L^i P_{\mathcal{R}})(P_{\mathcal{R}} L^i P_{\mathcal{R}}) = \frac{1}{2}(n^2 + 4n + 3)\\ &= \left(\frac{R}{l}\right)^2 = J\,.\end{aligned} \tag{D.12}$$

In sum, the $J \times J$ matrices are filled up with $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ representations which are of the form $\mathcal{R} \otimes \mathcal{R}$ where $\mathcal{R}$ is given in (D.10), and size of the matrices, J and radius of the S^3_F are related as given in (D.11) and (D.12). As an example, for the lowest J, $J = 4$, the above simply reduces to using the Γ matrix basis for 4×4 matrices, i.e. $\mathbf{1}$, Γ^5, Γ^i, $\Gamma^i\Gamma^5$, $\frac{1}{2}\Gamma^{ij}(1 \pm \Gamma^5)$.

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